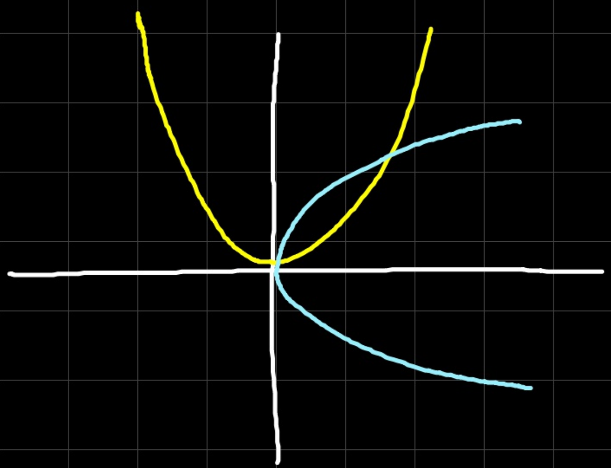
Inverse Functions

$y = x^2$ $\qquad$ $x = y^2$



f and g are inverses
iff
$$f(g(x)) = x \quad \forall x$$
and $g(f(x)) = x \quad \forall x$.

$$f(f^{-1}(x)) = x \text{ and}$$
$$f^{-1}(f(x)) = x$$

Show $f(x) = 3x - 2$ and $g(x) = \frac{x+2}{3}$ are inverses.

$$f(g(x)) = f\left(\frac{x+2}{3}\right)$$
$$= 3\left(\frac{x+2}{3}\right) - 2$$
$$= x$$

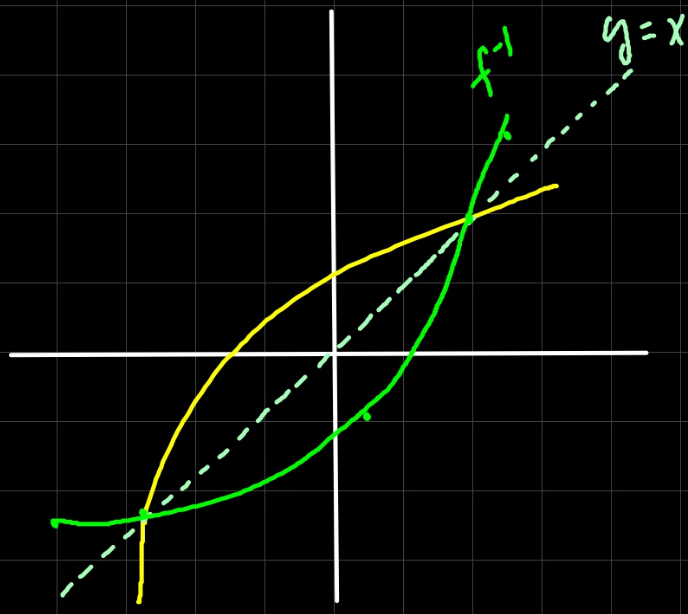
$$g(f(x)) = g(3x-2)$$
$$= \frac{(3x-2)+2}{3}$$
$$= x$$

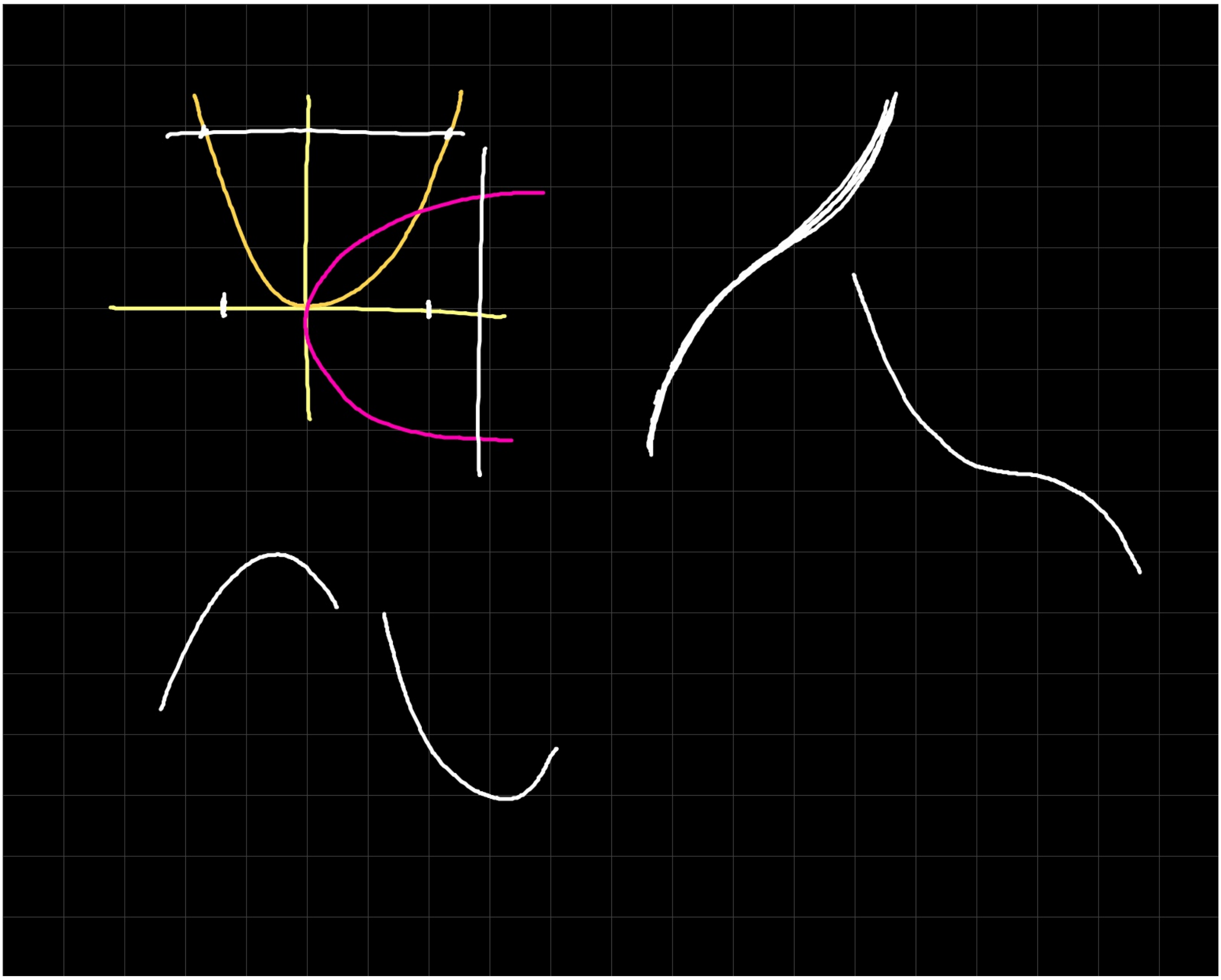
Since $f(g(x)) = x \ \forall x$
and $g(f(x)) = x \ \forall x$
f and g are inverse.

Notation

$x = f^{-1}(y)$

$y = f^{-1}(x)$





f has an inverse only if f is one-to-one (monotonic)

f is monotonic (has an inverse) iff

$f'(x) \geq 0 \ \forall x$ in f

or if

$f'(x) \leq 0 \ \forall x$ in f

Determine if $f(x) = \frac{x-2}{x+3}$ has an inverse.

$$f'(x) = \frac{5}{(x+3)^2}$$

Since $f'(x) \geq 0$ $\forall x$ in f

f has an inverse.

Find the inverse of $f(x) = 7x - 2$.

→ Let $y = f(x)$

$$y = 7x - 2$$

$$x = \frac{y+2}{7}$$

→ $f^{-1}(y) = \frac{y+2}{7}$

→ $f^{-1}(x) = \frac{x+2}{7}$

Find inverse of $f(x) = \frac{x+2}{x-3}$

let $y = f(x)$

$$y = \frac{x+2}{x-3}$$

$$y(x-3) = x+2$$

$$xy - 3y = x+2$$

$$xy - x = 3y+2$$

$$x(y-1) = 3y+2$$

$$x = \frac{3y+2}{y-1}$$

$$f^{-1}(y) = \frac{3y+2}{y-1}$$

$$\therefore f^{-1}(x) = \frac{3x+2}{x-1}$$

Show that $f(x) = x^5 + 5x^3 + 2x - 4$ has an inverse.

$f'(x) = 5x^4 + 15x^2 + 2$

Since $f'(x) \geq 0 \ \forall x$ in f, f has an inverse.

$(1, 4)$

Find the inverse

$(4, 1)$ on f^{-1}

$$y - y_1 = m(x - x_1) \qquad f \text{ @ } (c, d)$$

$$y - d = f'(c)(x - c)$$

$$\frac{y-d}{f'(c)} = x - c$$

$$x - c = \frac{1}{f'(c)}(y - d)$$

$$y - c = \frac{1}{f'(c)}(x - d)$$

$$\boxed{\text{If } (c,d) \text{ is on } f \text{ then } f^{-1}{}'(d) = \frac{1}{f'(c)}}$$

Write eq of tan to the inverse of

$f(x) = x^5 + 5x^3 + 2x - 4$ at $\underline{x=4}$.

$$x^5 + 5x^3 + 2x - 4 = 4 \longrightarrow x = 1$$

Since $(1,4)$ is on f then $(f^{-1})'(4) = \frac{1}{f'(1)}$

$f'(x) = 5x^4 + 15x^2 + 2 \longrightarrow f'(1) = 22 \longrightarrow (f^{-1})'(4) = \frac{1}{22}$

We know $(4,1)$ on f^{-1}

$\therefore\ y - 1 = \frac{1}{22}(x-4)$

Given $f(2)=5$ and $f'(2)=7$

and $g(x)=f^{-1}(x)$, find $g'(5)$.

$(f^{-1})'(5)$

Since $(2,5)$ on $f \rightarrow (f^{-1})'(5) = \dfrac{1}{f'(2)} = \dfrac{1}{7}$.