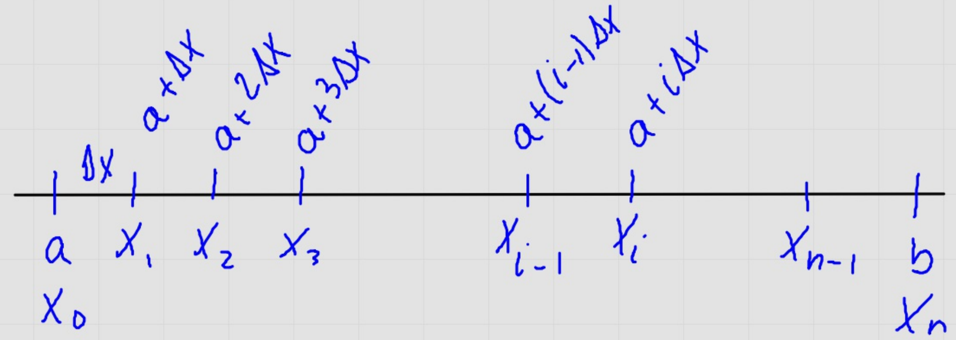
Exact Area via the Limit of a Riemann Sum



$$A_R \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

$$A_L \approx \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$\approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x$$

where $c_i = x_i = a + i\Delta x$ R

$c_i = x_{i-1} = a + (i-1)\Delta x$ S

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Bounds: $f(x) = 5x + 1$, $x = 2$, $x = 6$, x-axis.

$$\Delta x = \frac{4}{n} \qquad x_i = 2 + i\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(2 + i\Delta x)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[11\Delta x + 5i(\Delta x)^2\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{44}{n} + \frac{80}{n^2} i\right]$$

$$= \lim_{n\to\infty} \left[44 + \frac{80}{n^2} \frac{n^2+n}{2}\right]$$

$$= 44 + 40(1) = 84$$

Bonus: $f(x) = x^2$, $x = 1$, $x = 3$, x-axis

$$\Delta x = \frac{2}{n} \qquad x_i = 1 + i\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(1 + i\Delta x)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\Delta x + 2i(\Delta x)^2 + i^2(\Delta x)^3\right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{2}{n} + \frac{8}{n^2} i + \frac{8}{n^3} i^2\right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{8}{n^2}\frac{n^2+n}{2} + \frac{8}{n^3}\frac{2n^3+3n^2+n}{6}\right]$$

$$= 2 + 4(1) + \frac{4}{3}(2)$$

$$= \frac{26}{3}$$

Bands: $f(x) = x^2 + x$, $x = 2$, $x = 5$, x-axis.

$$\Delta x = \frac{3}{n} \qquad x_i = 2 + i\Delta x$$

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(2 + i\Delta x)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[6\Delta x + 5i(\Delta x)^2 + i^2(\Delta x)^3 \right]$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{18}{n} + \frac{45}{n^2} i + \frac{27}{n^3} i^2 \right]$$

$$= \lim_{n\to\infty} \left[18 + \frac{45}{n^2} \frac{n^2+n}{2} + \frac{27}{n^3} \frac{2n^3+3n^2+n}{6} \right]$$

$$= 18 + \frac{45}{2}(1) + \frac{9}{2}(2)$$

$$= \frac{99}{2}$$