1. $u = x^2 + x + 1 \longrightarrow du = (2x+1)\ dx$

$$\begin{aligned}\int (2x+1)(x^2+x+1)^3\,\mathrm{d}x &= \int u^3\,\mathrm{d}u \\ &= \frac{1}{4}u^4 + C \\ &= \frac{1}{4}(x^2+x+1)^4 + C\end{aligned}$$

2. $u = e^x \longrightarrow du = e^x\ dx$

$$\begin{aligned}\int e^x\ 2e^x\,\mathrm{d}x &= \int 2^u\,\mathrm{d}u \\ &= \frac{2^u}{\ln 2} + C \\ &= \frac{2^{e^x}}{\ln 2} + C\end{aligned}$$

3. $u = r^3 \longrightarrow du = 3r^2\ dr \longrightarrow \frac{1}{3}\ du = r^2\ dr$

$$\begin{aligned}\int r^2 \sec^2 r^3\,\mathrm{d}r &= \frac{1}{3}\int \sec^2 u\,\mathrm{d}u \\ &= \frac{1}{3}\ \tan u + C \\ &= \frac{1}{3}\ \tan r^3 + C\end{aligned}$$

4. This is a VERY important problem. If you ever face a rational function in which the degree of the numerator is greater than the degree of the denominator, reduce the complexity of the problem by dividing the denominator into the numerator.

$$\begin{aligned}\int \frac{1+x^3}{1+x}\,\mathrm{d}x &= \int (x^2 - x + 1)\,\mathrm{d}x \\ &= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x + C\end{aligned}$$

Note that in this problem, there was no remainder. If there were a remainder, use a u subsitution–the result generally involves the natural logarithmic function.

5. $u = \ln x \longrightarrow du = \frac{1}{x}\ dx$

$$\begin{aligned}\int \frac{\ln^2 x}{x}\,\mathrm{d}x &= \int u^2\,\mathrm{d}u \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\ln^3 x + C\end{aligned}$$

6. $u = e^x + 1 \longrightarrow du = e^x\, dx$

$$\int \frac{e^x}{e^x+1}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}u$$
$$= \ln|u| + C$$
$$= \ln|e^x+1| + C$$

7. $u = 2x \longrightarrow du = 2\, dx \longrightarrow \frac{1}{2}\, du = dx$

$$\int 3^{2x}\,\mathrm{d}x = \frac{1}{2}\int 3^u\,\mathrm{d}u$$
$$= \frac{3^u}{2\ln 3} + C$$
$$= \frac{3^{2x}}{2\ln 3} + C$$

8. $u = 2x - 1 \longrightarrow du = 2\, dx \longrightarrow \frac{1}{2}\, du = dx$

$$\int 8^{2x-1}\,\mathrm{d}x = \frac{1}{2}\int 8^u\,\mathrm{d}u$$
$$= \frac{8^u}{2\ln 8} + C$$
$$= \frac{8^{2x-1}}{2\ln 8} + C$$

9. $u = x^2 \longrightarrow du = 2x\, dx \longrightarrow \frac{1}{2}\, du = x\, dx$

$$\int x\, 5^{x^2}\,\mathrm{d}x = \frac{1}{2}\int 5^u\,\mathrm{d}u$$
$$= \frac{5^u}{2\ln 5} + C$$
$$= \frac{5^{x^2}}{2\ln 5} + C$$

10. $u = \sqrt{x} \longrightarrow du = \frac{1}{2\sqrt{x}} \longrightarrow 2\, du = \frac{1}{\sqrt{x}}\, dx$

$$\int \frac{5\csc\sqrt{x}}{\sqrt{x}}\,\mathrm{d}x = 10\int \csc u\,\mathrm{d}u$$
$$= 10\,\ln|\csc u - \cot u| + C$$
$$= 10\,\ln|\csc\sqrt{x} - \cot\sqrt{x}| + C$$

11. $u = x + 3 \longrightarrow du = dx$

$x = u - 3 \longrightarrow x + 1 = u - 2 \longrightarrow (x+1)^2 = u^2 - 4u + 4$

$$\begin{aligned}
\int \sqrt{x+3}\,(x+1)^2\,\mathrm{d}x &= \int u^{1/2}(u^2 - 4u + 4)\,\mathrm{d}u \\
&= \int \left(u^{5/2} - 4u^{3/2} + 4u^{1/2}\right)\mathrm{d}u \\
&= \frac{2}{7}u^{7/2} - \frac{8}{5}u^{5/2} + \frac{8}{3}u^{3/2} + C \\
&= \frac{2}{7}(x+3)^{7/2} - \frac{8}{5}(x+3)^{5/2} + \frac{8}{3}(x+3)^{3/2} + C \\
&= \frac{2\sqrt{(x+3)^7}}{7} - \frac{8\sqrt{(x+3)^5}}{5} + \frac{8\sqrt{(x+3)^3}}{3} + C
\end{aligned}$$

12. $u = 1 - 2x^2 \longrightarrow du = -4x\ dx \longrightarrow -\dfrac{1}{4}\ du = x\ dx$

From $u = 1 - 2x^2$ we have $x^2 = \dfrac{1-u}{2}$

$$\begin{aligned}
\int \frac{x^2 x}{\sqrt{1-2x^2}}\,\mathrm{d}x &= -\frac{1}{4}\int u^{-1/2}\left(\frac{1-u}{2}\right)\mathrm{d}u \\
&= -\frac{1}{8}\int \left(u^{-1/2} - u^{1/2}\right)\mathrm{d}u \\
&= -\frac{1}{8}\left[2u^{1/2} - \frac{2}{3}u^{3/2}\right] + C \\
&= -\frac{1}{4}\sqrt{1-2x^2} + \frac{1}{12}\sqrt{(1-2x^2)^3} + C
\end{aligned}$$

13. $u = x^3 \longrightarrow du = 3x^2\ dx \longrightarrow \dfrac{1}{3}\ du = x^2\ dx$

$$\begin{aligned}
\int x^2\ 10^{x^3}\,\mathrm{d}x &= \frac{1}{3}\int 10^u\,\mathrm{d}u \\
&= \frac{10^u}{3\ln 10} + C \\
&= \frac{10^{x^3}}{3\ln 10} + C
\end{aligned}$$

14. $u = x^4 + 2x \longrightarrow du = (4x^3 + 2)\ dx \longrightarrow \dfrac{1}{2}\ du = (2x^3 + 1)\ dx$

$$\begin{aligned}
\int 5^{x^4+2x}\ (2x^3+1)\,\mathrm{d}x &= \frac{1}{2}\int 5^u\,\mathrm{d}u \\
&= \frac{5^u}{2\ln 5} + C \\
&= \frac{5^{x^4+2x}}{2\ln 5} + C
\end{aligned}$$

15. $$\int \ln e^{\sin x}\,\mathrm{d}x = \int \sin x\,\mathrm{d}x$$
$$= -\cos x + C$$

16. $u = \sin x \longrightarrow du = \cos x\ dx$

$$\int (3\cos x) 8^{\sin x}\,\mathrm{d}x = 3\int 8^u\,\mathrm{d}u$$
$$= 3\,\frac{8^u}{\ln 8} + C$$
$$= 3\,\frac{8^{\sin x}}{\ln 8} + C$$

17. $u = x - 2 \longrightarrow du = dx$

$x = u + 2$

$$\int \frac{5x}{\sqrt{x-2}}\,\mathrm{d}x = 5\int u^{-1/2}(u+2)\,\mathrm{d}u$$
$$= 5\int \left(u^{1/2} + 2u^{-1/2}\right)\mathrm{d}u$$
$$= 5\left(\frac{2}{3}u^{3/2} + 4u^{1/2}\right) + C$$
$$= \frac{10\sqrt{(x-2)^3}}{3} + 20\sqrt{x-2} + C$$

18. $u = 8x - 1 \longrightarrow du = 8\ dx \longrightarrow \dfrac{1}{8}\ du = dx$

$$\int e^{8x-1}\,\mathrm{d}x = \frac{1}{8}\int e^u\,\mathrm{d}u$$
$$= \frac{1}{8}e^u + C$$
$$= \frac{1}{8}e^{8x-1} + C$$

19. $u = 1 + \sec x \longrightarrow du = \sec x \tan x\ dx$

$$\int \sec x \tan x \sqrt{1 + \sec x}\,\mathrm{d}x = \int u^{1/2}\,\mathrm{d}u$$
$$= \frac{2}{3}u^{3/2} + C$$
$$= \frac{2}{3}\sqrt{(1+\sec x)^3} + C$$

20. $$\int e^{\ln \csc x}\,\mathrm{d}x = \int \csc x\,\mathrm{d}x$$
$$= \ln|\csc x - \cot x| + C$$