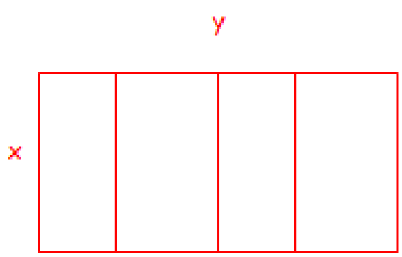
1. Let x be one of the numbers $\rightarrow$ $100-x$ is the other.
$P(x) = x(100-x)$
$P'(x) = 100 - 2x$
$P'(x) = 0$ when $x = 50$
$\therefore$ the numbers are 50 and 50.

2. Let x be one of the numbers $\rightarrow$ $\dfrac{100}{x}$ is the other.
$S(x) = \dfrac{100}{x} + x$
$S'(x) = \dfrac{x^2-100}{x^2}$
$S'(x) = 0$ when $x^2 - 100 = 0$ $\rightarrow$ $x = 10$ or $x = -10$
We choose $x = -10$ $\therefore$ the numbers are -10 and -10

3. Let x be the width and y be the length.
$P = 2x + 2y \rightarrow y = \dfrac{P-2x}{2}$
Now, $A = xy$ so $A(x) = x\left(\dfrac{P-2x}{2}\right) \rightarrow A(x) = \dfrac{P}{2}x - x^2$
$A'(x) = \dfrac{P}{2} - 2x$
$A'(x) = 0$ when $x = \dfrac{P}{4}$
Since $x = \dfrac{P}{4}$ the rectangle must be a square!

4. Long sides will be y and all short sides will be x.



Since the total fence to be used is 750, $5x + 2y = 750 \rightarrow y = \dfrac{750-5x}{2}$
Now, $A(x) = x\left(\dfrac{750-5x}{2}\right) \rightarrow A(x) = 375x - \dfrac{5}{2}x^2$
$A'(x) = 375 - 5x$
$A'(x) = 0$ when $x = 75 \rightarrow A(75) = 14062.500$ $\therefore$ the maximum area is 14062.500 square feet.

5. Let the length and width of the base be x and h be the height. $\rightarrow\ x^2+4xh=1200\ \rightarrow\ h=\dfrac{1200-x^2}{4x}$

   Now, $V(x)=x^2\left(\dfrac{1200-x^2}{4x}\right)\ \rightarrow\ V(x)=300x-\dfrac{1}{4}x^3$

   $V'(x)=300-\dfrac{3}{4}x^2\ \rightarrow\ V'(x)=0$ when $x=20$ or $x=-20$.

   Choosing $x=20\ \rightarrow\ V(20)=4000\ \therefore$ the maximum volume is 4000 cubic cm.

6. Let w be the width, $2w$ be the length, and h the height.

   Now, $V=2x^2h$ and since $V=10\ \rightarrow\ 10=2w^2h\ \rightarrow\ h=\dfrac{5}{w}$.

   We have a bottom of area $2w^2$, 2 sides of area $\dfrac{5}{w}$ and 2 sides of $\dfrac{10}{w}$.

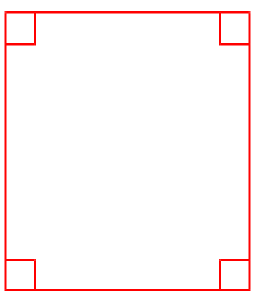
   The cost function then becomes $C(w)=10(2w^2)+6(2)\left(\dfrac{5}{w}\right)+6(2)\left(\dfrac{10}{w}\right)$

   Simplifying yields $C(w)=\dfrac{20w^3+180}{w}$.

   Now, $C'(w)=\dfrac{40w^3-180}{w^2}$

   $C'(w)=0$ when $40w^3-180=0\ \rightarrow\ w=1.651$.
   $\therefore$ the minimum cost occurs when $w=1.651\ \rightarrow\ C(1.651)=163.54\ \rightarrow$ the minimum cost is \$163.54.

7. If x is the length of the small cutout square, the the width becomes $3-2x$ and the length becomes $3-2x$ and the height of the box will be x.

   $V(x)=x(3-2x)^2\ \rightarrow\ V(x)=4x^3-12x^2+9x$

   $V'(x)=12x^2-24x+9\ \rightarrow\ V'(x)=0$ when $x=.500$ or $x=1.500$

   We choose $x=.5$ because $x=1.5$ creates an "unbox"!

   $V(.500)=2\ \therefore$ the maximum volume of the box is 2 cubic feet .

8. Any point on the curve can be described as $(x,2x-3)$.

   $D=\sqrt{(x-0)^2+(2x-3-0)^2}\ \rightarrow\ D=\sqrt{5x^2-12x+9}$

   Now, minimizing the square root of a number is equivalent to minimizing the number itself so we use

   $D(x)=5x^2-12x+9$.

   $D'(x)=10x-12\ \rightarrow\ D'(x)=0$ when $x=\dfrac{6}{5}\ \rightarrow\ y=-\dfrac{3}{5}$

   Thus, the point on $y=2x-3$ that is closest to $(0,0)$ is the point $\left(\dfrac{6}{5},-\dfrac{3}{5}\right)$

9. Any point on the curve can be described as $\left(x, \sqrt{4+x^2}\right)$.

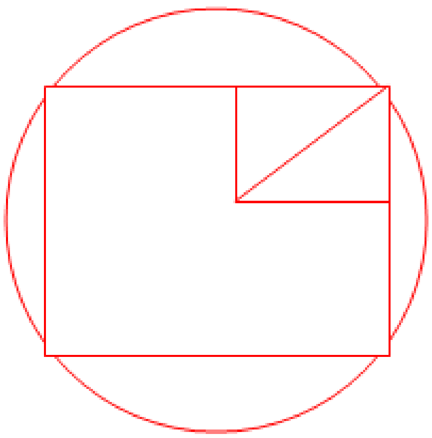
   Again, minimizing the square root of a number is equivalent to minimizing the number itself so we use

   $D(x) = (x-2)^2 + \left[\sqrt{4+x^2}\right]^2 \;\rightarrow\; D(x) = 2x^2 - 4x + 8.$

   $D'(x) = 4x - 4 \;\rightarrow\; D'(x) = 0$ when $x = 1 \;\rightarrow\; y = \sqrt{5}$ or $x = -\sqrt{5}$

   Thus, the points on $y^2 - x^2 = 4$ that are closest to $(2, 0)$ are the points $\left(1, \sqrt{5}\right)$ and $\left(1, -\sqrt{5}\right)$

10. Let x be half the height of the rectangle and y be half the width and r the radius of the circle … where r is a constant.



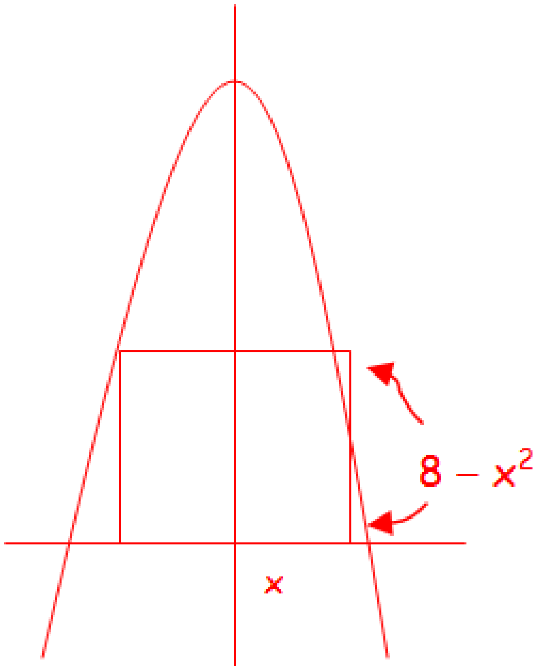
   $r^2 = x^2 + y^2 \;\rightarrow\; y = \sqrt{r^2 - x^2}$
   The area of the rectangle is $A = 4xy \;\rightarrow\; A(x) = 4x\sqrt{r^2 - x^2}$

   Now $A'(x) = \dfrac{4r^2 - 8x^2}{\sqrt{r^2 - x^2}} \;\rightarrow\; A'(x) = 0$ when $x = \dfrac{\sqrt{2}}{2}r$

   $\therefore$ the dimensions of the rectangle with the largest area are $r\sqrt{2}$ by $r\sqrt{2}$

   Note: Remember, x was half the height, so we need to multiply x by 2 to get the dimensions.

11. Any point on the parabola can be described as $(x, 8 - x^2)$



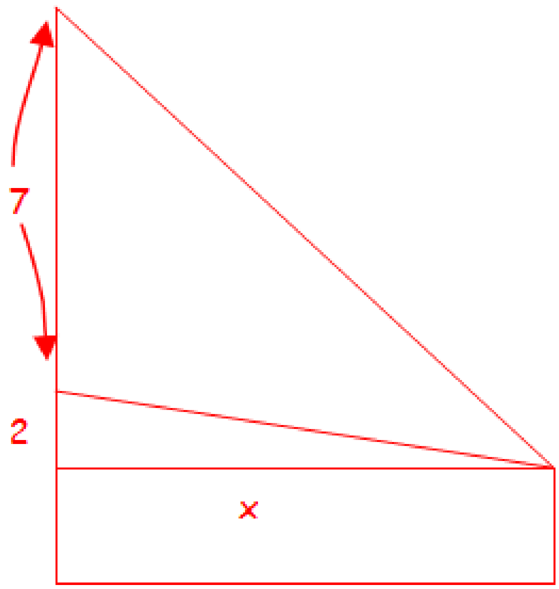
The area of one-half the rectangle is then $x(8 - x^2)$ so $A(x) = 16x - x^3$.

Now, $A'(x) = 16 - 6x^2 \rightarrow A'(x) = 0$ when $x = \dfrac{2\sqrt{6}}{3}$ or $x = -\dfrac{2\sqrt{6}}{3}$. We will use the positive value.

$\therefore$ the length of the base of the rectangle of maximum area is $\dfrac{4\sqrt{6}}{3}$ and the height is $\dfrac{16}{3}$

Note: Again, in this problem we let x be one-half the length of the base so we must multiply it by 2 to get the length of the base.

12. From the diagram below we will let the angle opposite the 7-foot picture be θ and the angle opposite the 2-foot level be α. We then let $\beta = \alpha + \theta$.



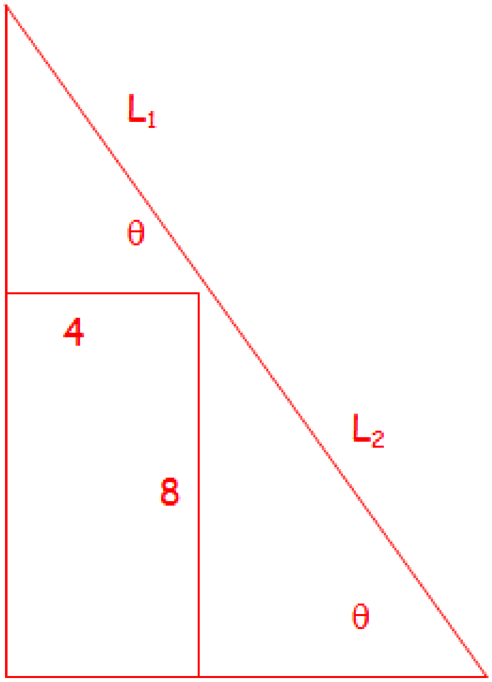
Since $\beta = \alpha + \theta \;\rightarrow\; \theta = \beta - \alpha$.

$\theta(x) = \tan^{-1}\dfrac{9}{x} - \tan^{-1}\dfrac{2}{x} \;\rightarrow\; \theta'(x) = \dfrac{126 - 7x^2}{x^4 + 85x^2 + 324}$

$\theta'(x) = 0$ when $x = -3\sqrt{2}$ or $x = 3\sqrt{2}$.

We take the positive value and therefore the person should stand $3\sqrt{2}$ feet from the wall...about 4.243 feet.

13. From the diagram below, $L = L_1 + L_2$.



$\sin\theta = \dfrac{8}{L_2} \;\rightarrow\; L_2 = 8\csc\theta$

$\cos\theta = \dfrac{4}{L_1} \;\rightarrow\; L_1 = 4\sec\theta$

Since $L = L_1 + L_2 \;\rightarrow\; L(\theta) = 8\csc\theta + 4\sec\theta$.

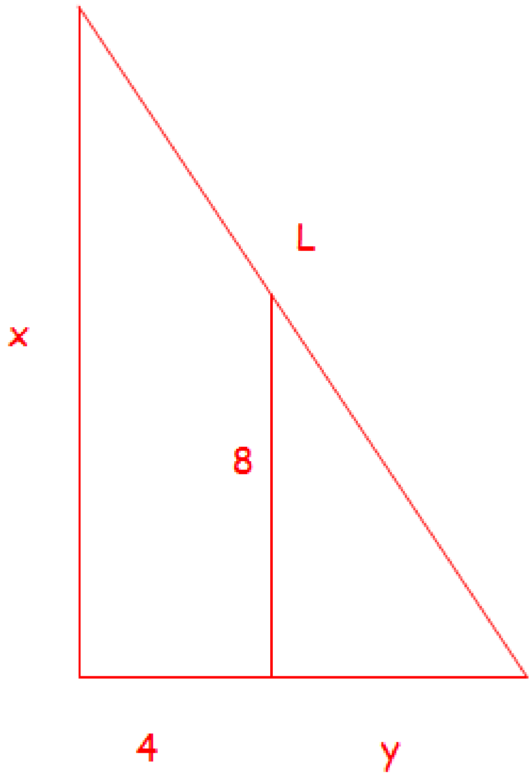
$L'(\theta) = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta$

$L'(\theta) = 0$ when $\theta = \tan^{-1}\sqrt[3]{2} \approx .900$

$L\left(\sqrt[3]{2}\right) \approx 16.648 \;\therefore$ the shortest ladder will have length 16.648.

An alternate solution using similar triangles can be found on the following page.

14. From similar triangles.



$$\frac{8}{y} = \frac{x}{y+4}$$
$$8y + 32 = xy$$
$$x = \frac{8y+32}{y}$$
$$\text{Now } L^2 = x^2 + (y+4)^2$$
$$L^2(y) = \left(\frac{8y+32}{y}\right)^2 + y^2 + 8y + 16$$
$$\left(L^2\right)'(y) = \frac{2(y^4 + 4y^3 - 256y - 1024)}{y^3}$$
$$\left(L^2\right)'(y) = 0 \text{ when } y = -4 \text{ or } y = 4\sqrt[3]{4} \;\rightarrow\; y = 4\sqrt[3]{4} \;\rightarrow\; x = 4\sqrt[3]{4}\left(1 + \sqrt[3]{4}\right)$$
$$\therefore L \approx 16.648$$